Linear Algebra MCV4U3

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1 Eigenvalues and Eigenvectors

1.1 Introduction

Recall the form of converting a conic into an easier conic.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

was converted into matrix form, otherwise known as

$$X^T AX + 2GX + C = 0$$

and was then rotated by way of a set of matrices $B = R_0 A R_0^{-1}$ to create

$$U^T BU + 2HU + C = 0$$

whereby we then completed the square, graphed, and drew the original.

We did this through a process of diagonalizing the matrix containing the slope values, A. The concept following is the same.

1.2 Taking Powers of Matrices

If a matrix is in the form

$$A = \begin{bmatrix} 3 & 7 \\ 2 & -1 \end{bmatrix}$$

then A^n is hard to find since we have to iterate through all of the powers. But if it is in the form

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

then it is easy to find, since we can simply take each diagonal value to the nth power. We can reach the final matrix product in one fell swoop. So B^n would be simply

$$\begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}$$

1.2.1 An Example Regarding Nuclear Reactions

Say a particular nuclear reaction has x and y particles. We want to find how many x and y particles exist after a certain period of time (in hours).

The relationship give to us, where k is the hour, forms a system of linear equations, below.

$$\begin{aligned} x_{k+1} &= x_k + 2y_k \\ y_{k+1} &= 3x_k + 2y_k \end{aligned}$$

This, rewritten, can actually be written as a linear matrix relation. This relation is

$$V_{k+1} = AV_k$$

Otherwise, written out, this becomes

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

Now from this, we realize that $V_1 = AV_0$ holds true. Furthermore, we realize that $V_k = A^k V_0$ also holds true. But herein lies the problem. A^k cannot be easily found without iterating through. Or can it?

If we diagonalize the matrix A, it will be easy to find A to a k^{th} power.

We can diagonlize the matrix much like we diagonalized the conic section equation. D is the diagonalized matrix in the following equations. The matrix P MUST be invertible.

$$\begin{aligned} D &= P^{-1}AP \\ PDP^{-1} &= A \\ A^2 &= (PDP^{-1})(PDP^{-1}) \\ A^2 &= PD^2P^{-1} \end{aligned}$$

Remember, D is

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

for some integer values a, b.

This brings us to the equation that will solve our problems:

$$A^n = PD^nP^{-1}$$

This is easy to solve!

1.3 Eigenvalues and Eigenvectors

We now first take a break from continuing this, and talk about what eigenvalues and eigenvectors are.

The definition is the following:

If matrix A is an nxn matrix, a number λ is called an eigenvalue of A if

$$AX = \lambda X$$

for some column vector $X \neq 0$. The vector X is the eigenvector for the eigenvalue λ.

This is interesting, since multiplying the two matrices on the left side yields a column matrix with the same dimensions, multiplied by a scalar.

As an example, lets see this in action.

$$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} 20 \\ 4 \end{bmatrix}$$

$$\text{but } 4 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix}$$

So X is an eigenvector of $\lambda = 4$ for the matrix A.

1.4 Finding Eigenvalues and Eigenvectors

We follow the steps below:

$$\begin{aligned} AX &= \lambda X \\ \lambda X - AX &= 0 \\ \lambda I X - AX &= 0 \\ (\lambda I - A)X &= 0 \end{aligned}$$

We also want to make sure we want the **non-trivial solution**, which means the solution vector is not $(0, 0)$ for some dimension n.

$$\begin{aligned} A &= \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda - 3 & -5 \\ 1 & \lambda + 1 \end{bmatrix} \end{aligned}$$

but in this case, remember that

$$\begin{bmatrix} \lambda - 3 & -5 \\ -1 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But remember, we want a non-trivial solution. But for the form $AX = 0$, if we want it to be nontrivial, then the matrix A cannot be linearly independent, and so it MUST be linearly dependent. Linear dependence implies $det(\lambda I - A) = 0$. This also means that if there are eigenvectors for a particular eigenvalue, then it is not simply a unique answer, there are many.

How do we know that the above is true? Here's why. Let's assume $(\lambda I - A)X = 0$ and for some resulting matrix

$$\begin{bmatrix} \lambda - a & -b \\ c & \lambda - d \end{bmatrix} = G$$

If $\lambda I - A$ has a determinant that isn't zero, then G is invertible.

$$\begin{aligned} GX &= 0 \\ X &= G^{-1}0 \\ X &= 0 \end{aligned}$$

But the trivial case is trivial.

We need to make the determinant of G equal to zero.

$$\lambda I - A = \begin{bmatrix} \lambda - 3 & -5 \\ -1 & \lambda + 1 \end{bmatrix}$$

$$\begin{aligned} det(\lambda I - A) = (\lambda - 3)(\lambda + 1) - (-1)(-5) &= 0 \\ 0 &= \lambda^2 - 2\lambda - 8 \\ 0 &= (\lambda - 4)(\lambda + 2) \end{aligned}$$

So $\lambda_1 = 4$ and $\lambda_2 = -2$. This means there are 2 eigenvalues. We substitute in the two values of λ into the matrix $\lambda I - A$ from before, to find eigenvectors.

For $\lambda = 4$

$$det(\lambda I - A) = \begin{bmatrix} (4) - 3 & -5 \\ 1 & (4) + 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix}$$

For $\lambda = -2$

$$det(\lambda I - A) = \begin{bmatrix} (-2) - 3 & -5 \\ 1 & (-2) + 1 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix}$$

Both of these produce a system of linear equations. For the case $\lambda = 4$, after reducing (or not at all in this case), we get

$$\begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This produces an equation

$$x - 5y = 0$$

with two variables, and so one parameter is introduced as usual, t, such that the general form of the eigenvector for this eigenvalue is

$$X_1 = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Similarly, for $\lambda_2 = -2$, the associated eigenvectors come in the form

$$X_2 = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For these, we may pick any of these to be used as part of diagonalization.

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

so

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & 0 & 0 \\ -1 & \lambda - 2 & 1 \\ -1 & -3 & \lambda + 2 \end{bmatrix}$$

Taking the determinant, we get values of $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$. Solving for each eigenvector by backsubstituting into $\lambda I - A$, we get

$$X_1 = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad X_2 = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad X_3 = t \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

1.5 Diagonalizing Matrices

If A is an nxn matrix:

Matrix A is diagonalizable if and only if it has eigenvectors $X_1, X_2, X_3, ..., X_k$ such that

$$P = \begin{bmatrix} X_1 & X_2 & X_3 & ... & X_k \end{bmatrix}$$

P is **not** a 1xk matrix. It is a matrix composed of the column vectors X_k.

And so

$$D = P^{-1}AP$$

$$V^k = A^k V_0$$

Since X_1 to X_k are linearly dependent, then P is invertible

$$\begin{aligned} D &= P^{-1}AP \\ &= diag(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k)] \\ &= diag(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k)] \\ &= \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & .. & \lambda_k \end{bmatrix} \end{aligned}$$

1.5.1 When Matrices Are Not Diagonalizable

For 2x2 matrices,

- If $\lambda_1 = \lambda_2$, then P is not invertible, and so A is not diagonalizable. This is a multiplicity of 2 for λ.
- If $\lambda_1 \neq \lambda_2$, it is diagonalizable since P is linearly independent.

$$P = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$$

For 3x3 matrices

- If λ_n are distinct values, it is diagonalizable,

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

- If λ_1 is distinct, but $\lambda_2 = \lambda_3$, then we have a subcase. We then look at the $\lambda I - A$ matrix.
  - Case 1:

    $$\lambda I - A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$$

    so that there are 2 distinct equations, and only one parameter, it is not diagonalizable, since it produces only X_2.

    $$P = \begin{bmatrix} X_1 & X_2 & X_2 \end{bmatrix}$$

  - Case 2:

    $$\lambda I - A = \begin{bmatrix} x & x & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

    so that there is 1 distinct equation, and only two parameters, it is diagonalizable, and produces X_2 and X_3.

    $$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$$

- If $\lambda_1 = \lambda_2 = \lambda_3$, then **at most** we can get two eigenvectors X_2 and X_3, similar to the case above.

1.6 Complex Eigenvalues

Recall how we can rotate vectors using rotation matrices.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

This effectively rotates the original vector (x, y) 90 degrees CCW.

If we have an equation of

$$\lambda^2 + 1 = 0$$

, then the roots are imaginary, notably: $\lambda_1 = i$ and $\lambda_2 = -i$

Examples are the following:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Just like regular numbers, eigenvalues also come in conjugates. So do the Matrices.

If

$$Z = \begin{bmatrix} -i+2 & 5 \\ i & 3+4i \end{bmatrix}$$

Then

$$\bar{Z} = \begin{bmatrix} i+2 & 5 \\ -i & 3-4i \end{bmatrix}$$

Also, If

$$X = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

So

$$Re(X) = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

And

$$Im(X) = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

Example:

Let A be

$$\begin{bmatrix} \frac{1}{2} & \frac{-3}{5} \\ \frac{3}{4} & \frac{11}{10} \end{bmatrix}$$

We can find the characteristic polynomial as $\lambda^2 - 1.6\lambda + 1 = 0$, so the roots are $\lambda_{1,2} = 0.8 \pm 0.6i$

We somehow then use $A - \lambda I$ instead, and again doing the usual substitution of λ into the matrix, we get matrices. However, we can't solve them using a graphing calculator. We CAN however do this by hand by building linear equations from our matrix! After all, matrices are just linear equations. Using the $\lambda_1 = 0.8 - 0.6i$ as an example, we get:

$$(-0.3 + 0.6i)x - 0.6y = 0$$
$$0.75x + 90.3 + 0.6i)i = 0$$

We combine, simplify, and substitute a value of y that gives integers. An example is $y = 5$, producing

$$X_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$$

As it turns out, the other eigenvector obtained from the other root is similar:

$$X_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$$

Conjugates! Realize that rotation matrices from before have eigenvalues built in.

Also, now we use A as a rotation matrix, and to diagonalize. $D = P^{-1}AP$ Remember how previously D gave a diagonal matrix with eigenvalues across the diagonal. Now lookie here.

But now, P is different. We use:

$$P = \begin{bmatrix} Re(X_1) & Im(X_1) \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$$

If we actually multiply out D, what a surprise, our D looks like:

$$D = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Well. That looks a bit like a rotation matrix! Lets visualize rotating a point (a, b), distance r from the origin in the Real-Imaginary space. the D matrix is then:

$$D = r \begin{bmatrix} \frac{a}{r} & \frac{-b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix} = r \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Exciting.

Oh but then think about it. What makes these eigenvectors special that they trace out elliptical (not circular) points? Their modulus, $|\lambda|$ is 1. If it wasn't 1, i.e. bigger or smaller, they trace out expanding or shrinking spirals.

1.7 Applications to Chaining

Going back to nuclear reactions, recall

$$V_{k+1} = AV_k$$
$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

Consider the dynamical system with matrix recurrence $V_{k+1} = AV_k$ for $k \geq 0$. Assuming that A is diagonalizable nxn matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, and corresponding eigenvectors $X_1, X_2, ..., X_n$, then an exact formula for V_k can be found. This is not the off by 1 formula which is rather: get to 10, do multiplying 9 times.

Instead of iterating with P, P^{-1}, D, we use

$$V_k = b_1\lambda_1^k X_1 + B_2\lambda_2^k X_2 + b_3\lambda_3^k X_3 + ... + b_n\lambda_n^k X_n$$

The coefficient of b_i comes from

$$P^{-1}V_0 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ ... \\ b_n \end{bmatrix}$$

Since V_k is a matrix, we take the equation that has x_k on one side, not, for example, the one with x_{k+1}.

This can be proven to work with even the fibonnacci sequence! The sequence is $1, 1, 2, 3, 5, 8, 13, ...$ and so on.

We set up

$$\begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

Once again, we do the operation $\lambda - IA$, take the determinant, set to zero, and solve. To find eigenvectors we substitute eigenvalues into our $\lambda - IA$ and use that as $(\lambda - IA)X = 0$ and solve for X_n, our eigenvectors.

The characteristic polynomial is $\lambda^2 - \lambda - 1$, eigenvalues of $\lambda_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$. Hmm. This is $\pm\Phi$ isn't it.

$$X_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

$$P^{-1}V_0 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

So that

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = b_1 \lambda_1^k X_1 + B_2 \lambda_2^k X^2$$

Becomes

$$x_k = \frac{1}{\sqrt{5}}[\lambda_1^{k+1} - \lambda_2^{k+1}]$$

Test it out. The positive eigenvalue 1.618... is dominant. It matches the pattern for the fibonnaci at any n^{th} term.

If $k = 12$, then

$$\begin{aligned} x_{12} &= \frac{(1.618)^{13}}{\sqrt{5}} \\ &= 2.32.94 \\ &= \ 233 \end{aligned}$$

Notice how we used only the dominant eigenvalue for this calculation. If we used both values, the term would be an integer, as expected. So this means for the limit as x approaches infinity, the effect of the dominant eigenvalue increases, and other values decrease to zero.

Note: for equations of three variables, we can set it up like this:

$$\begin{bmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$$

Also, if the dominant eigenvalue, i.e. the largest $|\lambda|$ is occurs on an eigenvalue of $\lambda = 1$, then the matrix will stabilize. If the value is above 1, it will increase forever, and if it is below 1, it will decrease to zero.

1.7.1 Predator/Prey Exercises

$$x_{n+2} = 5x_{n+1} + 6x_n$$

$$x_{n+2} - 5x_{n+1} - 6x_n = 0$$

Hmm. The characteristic equation is $r^2 - 5r - 6 = 0$, like second order differential equations, $y'' - 5y' - 6y = 0$. Guess what. They work because of eigenvalues, except no one explains it like that.

Single Species
a_k= number of adult females
j_k= number of juveniles
k = number of years
Starting conditions: 100 adult, 60 juvenile.

- The number of juvenile females hatched in a year is twice the number of adult females ... reproduction rate is 2
- Half of the adult females in any year survive to the next year. Survival rate is $\frac{1}{2}$

- One quarter of juvenile survive to adulthood.

So we have

$$a_{k+1} = \frac{1}{2}a_k + \frac{1}{4}j_k$$
$$j_{k+1} = 2a_k$$

So we get

$$\begin{bmatrix} a_{k+1} \\ j_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a_k \\ j_k \end{bmatrix}$$
$$0 = (\lambda - 1)(\lambda + \frac{1}{2})$$

$$\lambda_1 = 1 \qquad\qquad \lambda_2 = -\frac{1}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad\qquad X_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 220 \\ -80 \end{bmatrix}$$

$$\begin{bmatrix} a_k \\ j_k \end{bmatrix} = b_1\lambda_1^k X_1 + B_2\lambda_2^k X^2$$
$$\begin{bmatrix} a_k \\ j_k \end{bmatrix} = \frac{220}{3}(1)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{80}{3}(-\frac{1}{2})^k \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

So as the limit approaches infinity, $a_k = 73.33$ and $j_k = 146.66$. This is because of the dominant eigenvalue of 1. Again, if the dominant eigenvalue is less than 1, the population in question will "go the way of the dodo".

In this case, the dominant eigen values are found by multiplying $\frac{220}{3} * 1 * 1$ for a_k, and $\frac{220}{3} * 1 * 2$ for j_k, of which the values come from the numbers associated with the dominant eigenvalue.

If the table given is the following, with the question:

Determine if the population stabilizes, becomes extinct, or becomes large in each case. Denote the adult and juvenile survival rates as A and J, and the reproduction rate as R.

R	A	J
2	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{4}$	$\frac{1}{4}$
2	$\frac{1}{4}$	$\frac{1}{3}$
3	$\frac{3}{5}$	$\frac{1}{5}$

Then what this really means is this:

$$a_{k+1} = Aa_k + Jj_k$$
$$j_{k+1} = Ra_k$$

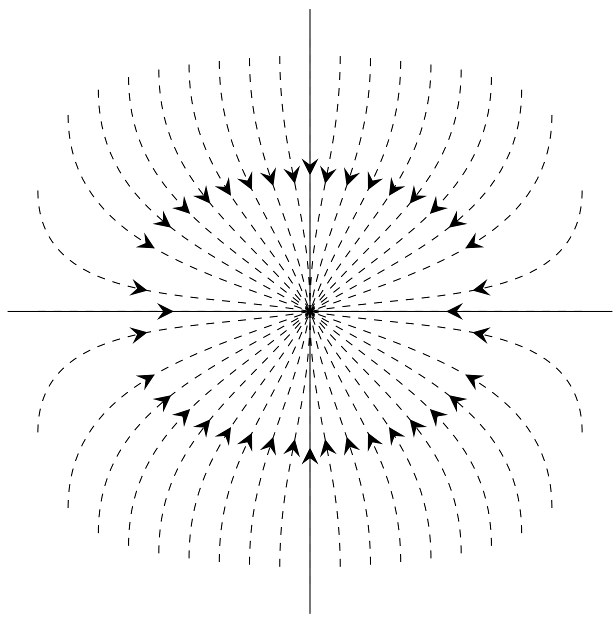
1.8 Graphical Descriptions of Solutions

In R^2 space, patterns may emerge when following a set of vectors transformed by a 2x2 matrix A.

Example:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}$$
$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$X_k = c_1(0.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Following a series of vectors through, the graph producted by the vectors looks somewhat like this:



As you see, the vectors curve towards the x-axis almost like an asssymptote. This is because both eigenvalues are < 1, which means that as limit approaches infinity, vectors decrease to zero. The curve is because one approaches 0 faster than the other.

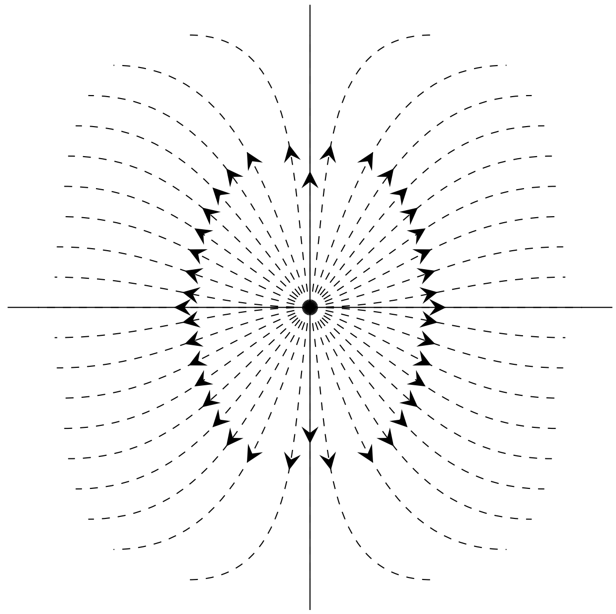
Along the lines $y = 0$ and $x = 0$ however, vectors don't curve. This makes sense as these are multiples of the basis eigenvector used, which are $(0, 1)$ and $(1, 0)$. In this sense, eigenvectors define lines where vectors transformed on this line map only to another portion of this line.

Notice that in all cases, these vectors pass through $(0, 0)$.

Let's look at another example.

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$
$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$X_k = c_1(1.44)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The resulting vector graph is

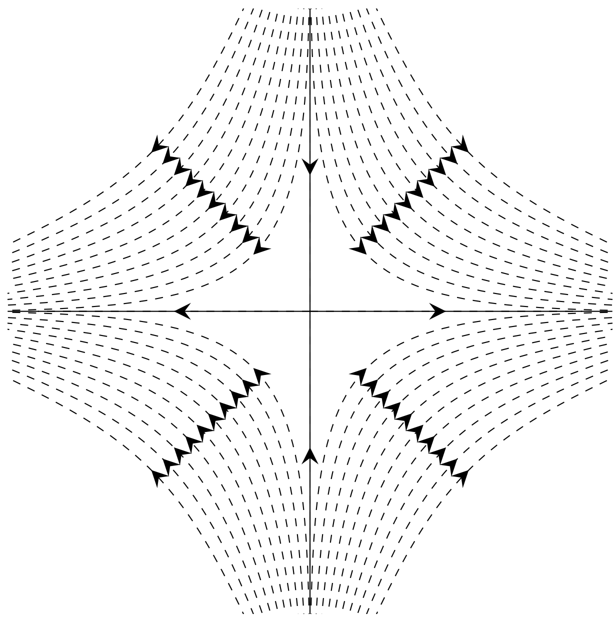


As you can see, the vectors tend to head horizontally outwards. Since both eigenvalues are larger than one, this graph will increase as limit approaches infinity. However, since one eigenvalue is bigger than the other, vectors will grow bigger in one direction than the other. It decreases to zero along x axis, and also decreases to zero along y axis.

Let's look at another example.

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \\
X_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
X_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
X_k &= c_1(2)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

The resulting vector graph is

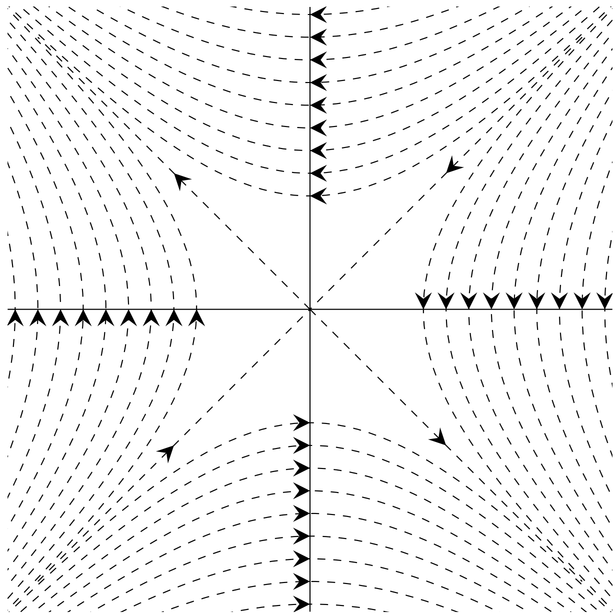


In this example, the two axis (acting like assymptotes) are $(1, 0)$ and $(0, 1)$. Since 2 modifies the x axis, it increases to infinity as 2 ¿ 1. Also, 0.5 ¡ 1, so the y axis assymptote decreases to zero.

One final example:

$$A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$$
$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$X_k = c_1(2)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(0.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Even though the eigenvectors arent the coordinate axes anymore, it still works the same way.



To summarize, the graphical effect of eigenvectors is that for vectors that are a multiple of eigenvectors:

$$T :\to X_n = x(X_n)$$

1.9 Graphical Descriptions for Complex eigenvectors

One more example involving complex eigenvalues is the following:

$$A = \begin{bmatrix} 0.8 & 0.5 \\ -0.1 & 1.0 \end{bmatrix}$$
$$\lambda_{1,2} = 0.9 \pm 0.2i$$
$$X_{1,2} = \begin{bmatrix} 1 \pm 2i \\ 1 \end{bmatrix}$$